MUTUAL STATIONARITY AND COMBINATORICS AT $\aleph_\omega$

WILLIAM ADKISSON AND DIMA SINAPOVA

ABSTRACT. Mutual stationarity for $\langle \kappa_n \mid n < \omega \rangle$ says that for any stationary sequence $S_n \subset \kappa_n$ and any algebra on $\sup_n \kappa_n$, there is a simultaneous witness for stationarity i.e. an elementary substructure M such that for all n, $\sup(M \cap \kappa_n) \in S_n$. We prove that mutual stationarity for $\langle \aleph_n \cap \mathrm{cof}(\omega_k) \mid k < n < \omega \rangle$ is consistent with the tree property at $\aleph_{\omega+1}$. Our second theorem is that mutual stationarity for $\langle \aleph_n \cap \mathrm{cof}(\omega_k) \mid k < n < \omega \rangle$ is consistent with the failure of SCH at $\aleph_\omega$. Both theorems use large cardinal hypotheses.

1. Introduction

Stationary sets are a fundamental notion in modern set theory. They are related to elementary substructures by identifying clubs as algebras as follows. For a regular cardinal κ and $\kappa < \lambda$, $S \subset \kappa$ is stationary iff for every algebra $\mathfrak{A}$ on λ, there is an elementary $N \prec \mathfrak{A}$, such that $\sup(N \cap \kappa) \in S$. This notion has an analogue for singular cardinals, called *mutual stationarity*.

Mutual stationary was introduced in 2001 by Foreman and Magidor in [7], and was used to show the nonsaturation of the nonstationary ideal on $\mathcal{P}_{\omega_1}(\lambda)$. Here is the definition:

Definition 1.1. Let R be a set of uncountable regular cardinals and $S = \langle S_\kappa \mid \kappa \in R \rangle$ be a sequence of stationary sets with $S_\kappa \subseteq \kappa$. The sequence S is mutually stationary if for every algebra $\mathfrak{A}$ on $\sup(R)$ there is $M \prec \mathfrak{A}$ such that $\sup(M \cap \kappa) \in S_\kappa$ for every $\kappa \in R \cap M$.

Suppose now that R consists of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ with limit κ. Given $A_n \subset \kappa_n$, we say that **mutual stationarity holds at** $\langle A_n \mid n < \omega \rangle$ if every sequence of stationary sets $S_n \subset A_n$ is mutually stationary.

Restricting to countable cofinality, in [7] Foreman and Magidor showed that mutual stationarity holds for $\langle \kappa_n \cap \mathrm{cof}(\omega) \mid n < \omega \rangle$. On the other hand, they showed that this result does not generalize to higher fixed cofinality. In particular, in L there is a sequence of stationary sets $S_n \subset \aleph_n \cap \mathrm{cof}(\omega_1)$, $n > 1$, which is not mutually stationary. This prompted the question of whether it is consistent to have mutual stationarity at the $\aleph_n$'s for higher fixed cofinality.

Since then there has been a long line of results on this topic. It turns out that mutual stationarity for uncountable cofinality both follows from large cardinals and has large cardinal strength. Here are some highlights:

(1) If each κ_n is supercompact, then every sequence of stationary sets $S_n \subset \kappa_n$ is mutually stationary.

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(2) (Cummings-Foreman-Magidor) [4] If $\mathbb{P}$ is the Prikry forcing to singularize κ and $\langle \kappa_n \mid n < \omega \rangle$ is the corresponding Prikry sequence, then in $V[\mathbb{P}]$ every sequence of stationary sets $S_n \subset \kappa_n$ is mutually stationary.
(3) (Koepke) [9] From a measurable cardinal, one can force mutual stationarity for $\langle \aleph_{2n+1} \cap \mathrm{cof}(\omega_1) \mid 1 < n < \omega \rangle$
(4) (Koepke-Welch) [10] A measurable cardinal is necessary to obtain mutual stationarity for $\langle \kappa_n \cap \mathrm{cof}(\omega_1) \mid n < \omega \rangle$, where $\langle \kappa_n \mid n < \omega \rangle$ are increasing regular cardinals.

Still, for a long time Foreman and Magidor's original question remained open. Then in 2019, Ben Neria [2] gave a positive answer. More precisely, he showed that from ω supercompacts it is consistent that every sequence of stationary sets $S_n \subseteq \omega_n$ of some fixed cofinality is mutually stationary. His model is obtained by forcing with Levy collapses to make the supercompacts be the $\aleph_n$'s. In Ben Neria's model, SCH holds at $\aleph_\omega$ (and actually GCH is true). Moreover, approachability at $\aleph_\omega$ holds, and all known models of the tree property at $\aleph_{\omega+1}$ have failure of approachability at $\aleph_\omega$. This raises the questions whether mutual stationarity at the $\aleph_n$'s for a fixed uncountable cofinality is consistent with the failure of SCH; and whether it is consistent with the tree property. In this paper we show the answer to both questions is yes.

Theorem 1.2. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ are ν^+-supercompact cardinals, where $\nu = \sup_n \kappa_n$. Then there is a forcing extension where for all $k < \omega$, mutual stationarity holds for $\langle \aleph_n \cap \mathrm{cof}(\omega_k) \mid k < n < \omega \rangle$ and the tree property holds at $\aleph_{\omega+1}$.*

Theorem 1.3. *Suppose that $\kappa < \mu < \lambda$ are supercompact cardinals. Then there is a forcing extension where for all $k < \omega$, mutual stationarity holds for $\langle \aleph_n \cap \mathrm{cof}(\omega_k) \mid k < n < \omega \rangle$ and SCH fails at $\aleph_\omega$.*

The first theorem, together with Ben Neria's model, shows that mutual stationarity and the tree property are in a sense orthogonal.

The motivation for the second theorem is that the failure of SCH is an instance of incompactness, since it requires small powerset below a singular κ and large powerset at κ. In contrast, mutual stationarity can be viewed as a compactness type principle, as it is similar in spirit to stationary reflection and follows from large cardinals. In addition, a corollary of this theorem is that one can reduce the large cardinal assumption of Ben-Neria's result in [2].

The paper is organized as follows. In section 2 we go over some preliminaries and facts which will be used to prove mutual stationarity. In section 3 we prove Theorem 1.2. Then in section 4 we prove Theorem 1.3.

2. Obtaining Mutual Stationarity From Ideals

In this section we summarize techniques due to Ben Neria [2] we will use throughout this paper to prove mutual stationarity. For a more detailed exposition of these techniques, see [2, Section 2]. Note that [2] uses the Jerusalem forcing convention, which this paper does not.

Definition 2.1. Suppose $M \prec \mathfrak{A}$. We call an extension N of M an *end-extension of M above λ* if $M \prec N \prec \mathfrak{A}$ and $N \cap \lambda = M \cap \lambda$.

To show that a sequence is mutually stationary, we will work inductively, starting with M_n and producing an end-extension M_{n+1}. The following standard result shows that it is enough to verify mutual stationarity on a tail, so we can start this process at any finite stage n.

Fact 2.2. *[7, Lemma 23] Let ν be a regular cardinal less than the least element of a set of regular cardinals K. If $\{S_\kappa \mid \kappa \in K\}$ is mutually stationary, and for all κ, $S_\kappa \subseteq \mathrm{cof}(\leq \nu)$, then for all $\lambda_1, \ldots, \lambda_n$ greater than ν and not in K, and all sequences of stationary sets $S_{\lambda_i} \subseteq \lambda_i \cap \mathrm{cof}(\leq \nu)$, the sequence $\{S_\kappa \mid \kappa \in K\} \cup \{S_{\lambda_i}, \ldots S_{\lambda_n}\}$ is mutually stationary.*

End extensions will be constructed via ideals.

Definition 2.3. A nonprincipal κ-complete ideal I on κ is μ-closed if I^+ has a $\leq_I$-dense subset D such that the $\leq_I \restriction D$ is μ-closed, i.e. closed under $< \mu$-sequences.

An ideal on κ is *nonstationary* if it extends the nonstationary ideal.

Lemma 2.4. *[2, Proposition 2.12] Suppose $\mu < \kappa$ are regular cardinals and $\mathfrak{A}$ is an algebra extending $\langle H_\theta, \in, <_\theta\rangle$ for some regular cardinal $\theta > 2^\kappa$. Let $M \prec \mathfrak{A}$ be a substructure of size μ closed under sequences of size $< \mu$, and let $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ be a stationary subset of κ in M. Suppose also that at least one of the following holds:*

(1) S consists of approachable points [1] or
(2) either κ is inaccessible or $\kappa = \tau^+$ and $\tau^{<\tau} = \tau$.

If S is positive with respect to some nonstationary κ-complete $(\mu+1)$-closed ideal on κ, then for every regular cardinal $\lambda \in M \cap \kappa$, there is a μ-closed substructure $N \prec \mathfrak{A}$ of size μ which is an end-extension of M above λ and satisfies $\sup(N \cap \kappa) \in S$.

Remark 2.5. For the proof in the case of the approachability assumption, see [2, Remark 2.9] for details.

To check that every sequence of stationary sets is mutually stationary, it suffices to show that these hypotheses are satisfied at each stage of the induction. Next, we define a principle that captures the key hypothesis of Lemma 2.4.

Definition 2.6. Let $\nu < \theta$ be uncountable cardinals. We say $\dagger^\nu_\theta$ holds if for all stationary $S \subset \theta$, there is a nonstationary θ-complete, $(\nu+1)$-closed ideal, for which S is a positive set. Given a poset $\mathbb{Q}$, we say that $\dagger_{\theta,\mathbb{Q}}$ holds if $1_\mathbb{Q}$ forces that for all uncountable ν with $\nu^{++} < \theta$, for all stationary $\dot{S} \subset \theta$, there is a nonstationary θ-complete, $(\nu+1)$-closed ideal, for which $\dot{S}$ is a positive set.

By the previous lemma, to ensure that mutual stationarity holds below $\aleph_\omega$ for sets of points of cofinality $\aleph_k$, it suffices to check that $\dagger^{\aleph_k}_{\aleph_n}$ holds for cofinitely many $n < \omega$ and that all relevant stationary sets are approachable. More precisely:

Lemma 2.7. *Suppose that for some $k < \omega$, for all large n, $\dagger^{\aleph_k}_{\aleph_n}$ holds and all stationary sets of $\aleph_n$ are approachable or GCH holds. Then mutual stationarity holds for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega\rangle$.*

Proof. Fix $k > 0$ and a stationary sequence $S_n \subset \aleph_n \cap \mathrm{cof}(\aleph_k)$, for $n > k$. Suppose that $\mathfrak{A}$ is an algebra on $\aleph_\omega$. Construct a sequence of elementary substructures of $\mathfrak{A}$, $\langle M_n \mid k < n < \omega\rangle$ by induction on n, as follows. Let M_{k+1} be such that $\sup(M_{k+1} \cap \aleph_{k+1}) \in S_{k+1}$. Now, suppose $n > k+1$ and we have defined M_{n-1}. By $\dagger^{\aleph_k}_{\aleph_n}$, there is a nonstationary $\aleph_n$-complete $(\aleph_k + 1)$-closed ideal I on $\aleph_n$ such that

[1] For the definition of approachable points see for example [3, Section 8]

$S_n \in I^+$. Then by Lemma 2.4, there is an elementary substructure M_n of $\mathfrak{A}$, such that M_n is an end extension of M_{n-1} above $\aleph_{n-1}$ and $\sup(M_n \cap \aleph_n) \in S_n$. Finally, let $M = \bigcup_n M_n \prec \mathfrak{A}$. Then for all $n > k$, $\sup(M \cap \aleph_n) = \sup(M_n \cap \aleph_n) \in S_n$. $\square$

Ideals as above are obtained from large cardinal embeddings.

Lemma 2.8. *[2, Fact 2.14] Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$. Let $\mathbb{P} \in V$ be a poset and let G be generic for $\mathbb{P}$. Suppose that $j(\mathbb{P})$ projects to $\mathbb{P}$, so that every $j(\mathbb{P})/G$ generic contains j"G. Working in $V[G]$, for every $\gamma \in j(\kappa) \setminus \kappa$ and $r \in j(\mathbb{P})/G$, define an ideal $I_{\gamma,r}$ by*

$$I_{\gamma,r} = \{\dot{X}_G \mid r \Vdash_{j(\mathbb{P})/G} \gamma \notin j(\dot{X})\}.$$

Then this ideal is well defined and has the following properties:

- *$I_{\gamma,r}$ is κ-complete and nonprincipal.*
- *$I_{\gamma,r}$ is nonstationary iff $r \Vdash \gamma \in j(\dot{C})$ for every $\mathbb{P}$-name $\dot{C}$ for a club subset of κ.*
- *If $j(\mathbb{P})/G$ is $(\mu + 1)$-closed for some $\mu < \kappa$, then $I_{\gamma,r}$ is a $(\mu + 1)$-closed ideal.*

Proof. We only briefly outline the proof. More details can be found in Foreman's chapter in the handbook [6], see the interlude to Section 7, "The Basic Idea".

First, note that if $\dot{X}_G = \dot{X}'_G$, then some condition in $j''G$ will force that $j(\dot{X}) = j(\dot{X}')$. Since any generic extension by $j(\mathbb{P})/G$ must contain $j''G$, any condition in $j(\mathbb{P})/G$ will force that $j(\dot{X}) = j(\dot{X}')$. It follows that the ideal is well-defined.

$I_{\gamma,r}$ is κ-complete because κ is the critical point of the embedding, and it is non-principal, because $\gamma \geq \kappa$. The second assertion of the lemma is clear.

The last claim follows from the fact that $j(\mathbb{P})/G$ induces a generic for the poset $(I^+_{\gamma,r}, \leq_{I_{\gamma,r}})$. For example, if j is derived from a κ-complete measure U, one can consider the following projection π from $j(\mathbb{P})/G$ below r, to the poset $(I^+_{\gamma,r}, \leq_{I_{\gamma,r}})$. Let $\gamma = [f_\gamma]_U$; for $q = [f_q]_U \in j(\mathbb{P})$, $q \leq r$, set $\pi(q) = \{f_\gamma(x) \mid f_q(x) \in G\}$. Clearly, $\pi(q)$ is a positive $I_{\gamma,r}$ set, since q forces that it is in the dual filter. In particular, $q \Vdash_{j(\mathbb{P})/G} \gamma \in j(\dot{\pi(q)}) := j(\{f_\gamma(x) \mid f_q(x) \in \dot{G}\})$; since $q \leq r$, r certainly can't force γ to not be in $j(\dot{\pi(q)})$, so $\pi(q) \notin I_{\gamma,r}$. Also, if $q' \leq q$, then $\pi(q') \subset \pi(q)$, so the map is order preserving.

Finally, we verify that π is indeed a projection. Suppose $Y = \dot{Y}_G \leq_{I^+_{\gamma,r}} \pi(q)$. We claim that (in V), $A := \{x \mid f_q(x) \not\Vdash f_\gamma(x) \notin j(\dot{Y})\} \in U$. If A is not in U, then its complement must be, so $q \Vdash_{j(\mathbb{P})/G} \gamma \notin j(\dot{Y})$. Note that the empty condition of $j(\mathbb{P})/G$ forces $\gamma \in j(\dot{\pi(q)}) \Leftrightarrow q \in j(\dot{G})$. It follows that the empty condition forces $\gamma \in j(\dot{\pi(q)}) \implies \gamma \notin j(\dot{Y})$. We conclude that $r \Vdash \gamma \notin j(\dot{Y}) \cap j(\dot{\pi(q)})$, so $Y \cap \pi(q) \in I_{\gamma,r}$. But $Y \leq_{I^+_{\gamma,r}} \pi(q)$ by assumption, so $Y \cap \pi(q) \in I^+_{\gamma,r}$, a contradiction. Since $A \in U$, we can define a condition $q' = [x \to q'_x]_U \leq q$, such that for all $x \in A$, $q'_x \Vdash f_\gamma(x) \in j(\dot{Y})$ and if $x \notin A$, $q'_x \perp q_x$. By density, one can find such a condition in $j(\mathbb{P})/G$. Then $\pi(q') = Y$. We conclude that π is a projection from $j(\mathbb{P})/G$ to $(I^+_{\gamma,r}, \leq_{I_{\gamma,r}})$, so a generic for $j(\mathbb{P})/G$ will induce a generic for $(I^+_{\gamma,r}, \leq_{I_{\gamma,r}})$. $\square$

To verify $\dagger^{\aleph_k}_{\aleph_n}$, we will need to use embeddings that give sufficiently closed quotients, and we will need to check that the ideals we produce are nonstationary and meet the requisite stationary set. To do so we will use the following lemma, which is implicit in [2].

Lemma 2.9. *Let $\lambda \geq 2^\kappa$, and let $j : V \to M$ be a λ-supercompactness embedding with critical point κ. Suppose $\mathbb{P}$ is a λ-cc poset with $(|\mathbb{P}|^{<\lambda})^\kappa = \lambda$, such that $\mathbb{P}$ and j meet the hypotheses of Lemma 2.8, and $\Vdash_{\mathbb{P}} j(\mathbb{P})/\dot{G}$ is $(\mu+1)$-closed. Let G be generic for $\mathbb{P}$ over V. Let $S \subset \kappa$ be a stationary set in $V[G]$. Then there is a condition r and ordinal γ such that the ideal $I_{\gamma,r}$ given by Lemma 2.8 is $(\mu+1)$-closed and nonstationary, and $S \in I^+_{\gamma,r}$.*

Proof. Since $\mathbb{P}$ is λ-cc, we can enumerate (possibly with repetitions) all $\mathbb{P}$-names for clubs in κ by $\vec{C} = \langle \dot{C}_i \mid i < \lambda \rangle$. Since j is a λ-supercompactness embedding, this sequence is contained in M. The sequence $j''\vec{C} = \langle j(\dot{C}_i) \mid i < \lambda \rangle$ will also be in M, and is a sequence of $j(\mathbb{P})$-names for clubs in $j(\kappa)$. It follows that the empty condition of $j(\mathbb{P})$ forces that $\dot{C}^* = \bigcap_{i<\lambda} j(\dot{C}_i)$ is a club in $j(\kappa)$.

Let $\dot{S}$ be a $\mathbb{P}$-name for S; $\dot{S}$ is forced to be stationary by some condition $p \in G$. Then the empty condition of $j(\mathbb{P})/G$ forces that $j(\dot{S})$ is stationary in $j(\check{\kappa})$. Then there is a condition $r \in j(\mathbb{P})/G$ and an ordinal $\gamma \geq \kappa$ such that $r \Vdash \check{\gamma} \in j(\dot{S}) \cap \dot{C}^*$. Let $I = I_{\gamma,r}$. By Lemma 2.8, we conclude that I is κ-complete, nonprincipal, nonstationary, and $(\mu+1)$-closed. Since $r \Vdash \check{\gamma} \in j(\dot{S})$, by the definition of $I_{\gamma,r}$, we have that $S \in I^+$ (and actually in the dual filter). □

We end this section with some lemmas about forcings that preserve †.

Lemma 2.10. *Suppose that $\nu < \kappa$ are cardinals, and $\dagger^\nu_\kappa$ holds in V. Let $\mathbb{P}$ be a poset in V, such that $|\mathbb{P}| \leq \nu$ and $\mathbb{P}$ preserves ν. Then if G is $\mathbb{P}$-generic, $\dagger^\nu_\kappa$ still holds in $V[G]$,*

Proof. Suppose that $S \subset \kappa$ is a stationary set in $V[G]$. Since $\mathbb{P}$ has size less than κ, there is a generic condition $p \in G$, such that $S_p = \{\alpha \mid p \Vdash^V_{\mathbb{P}} \alpha \in \dot{S}\}$ is stationary.

Let I_1 be the ideal given by $\dagger^\nu_\kappa$ in V applied to S_p. Now, going back to $V[G]$, let I be the ideal obtained from I_1. More precisely, $I = \{X \subset \kappa \mid \exists \bar{X} \in I_1, X \subset \bar{X}\}$.

First we show that I is a κ-complete ideal. Suppose that for some $\tau < \kappa$, $\langle X_i \mid i < \tau \rangle \in V[G]$ is a sequence of sets in I. Working in V, let $D_i = \{p \in \mathbb{P} \mid \exists \bar{X} \in I_1, p \Vdash \dot{X}_i \subset \bar{X}\}$. This is a dense subset of $\mathbb{P}$. For each $p \in D_i$, let $\bar{X}_{i,p} \in I_1$ witness membership. By κ-completeness of I_1, we have that $X := \bigcup_{i,p} \bar{X}_{i,p} \in I_1$. Since $\bigcup_i X_i \subset X$, we get that $\bigcup_i X_i \in I$.

Also, since $S_p \in I_1^+$, and $S_p \subset S$, we have that $S \in I^+$. Also if $A \subset \kappa$ is a nonstationary set in $V[G]$, since $|\mathbb{P}| < \kappa$, there is a nonstationary $A_1 \in V$ with $A \subset A_1$, and so $A \in I$.

So far we have shown that I is a κ-complete nonstationary ideal with S a positive set, using only that $|\mathbb{P}| < \kappa$.

Finally, let D be the $(\nu+1)$-closed dense subset of I_1^+. We claim that D remains a closed dense subset of I^+. For the density, we use that if X is an I-positive set in $V[G]$, by the κ-completeness of I_1 and $|\mathbb{P}| < \kappa$, there is $p \in G$, such that $X_p := \{\alpha \mid p \Vdash \alpha \in \dot{X}\} \in I_1^+$. Since D is dense there is $A \in D$ with $A \subset X_p \subset X$. For the closure, suppose that $\langle Y_i \mid i < \nu \rangle$ is a decreasing sequence of sets in D. Working in V, for each $i < \nu$ consider the dense set $E_i = \{q \in \mathbb{P} \mid \exists Y \in D, q \Vdash \check{Y} = \dot{Y}_i\}$, and for each $q \in E_i$, let $Y_{i,q} \in D$ be such that $q \Vdash Y_{i,q} = \dot{Y}_i$. Since $|\mathbb{P}| \leq \nu$ and D is $\nu+1$-closed, $\bigcap_{i,q \in E_i} Y_{i,q} \in D$. Since $\bigcap_{i,q \in E_i} Y_{i,q} \subset \bigcap_{i<\nu} Y_i$, we are done. □

Lemma 2.11. *Suppose that $\nu < \kappa$ are cardinals, and $\dagger^\nu_\kappa$ holds in V. Let $\mathbb{Q}$ be a poset in V, such that $|\mathbb{Q}| \leq \kappa$, and $\mathbb{Q}$ is $\leq \nu$-distributive (i.e. does not add sequences of size ν). Then if H is $\mathbb{Q}$-generic, $\dagger^\nu_\kappa$ still holds in $V[H]$,*

Proof. Suppose that $S \subset \kappa$ is a stationary set in $V[H]$. Since $|\mathbb{Q}| < \kappa$ by shrinking S, we may assume $S \in V$ and let I_1 be the ideal given by $\dagger^\nu_\kappa$ in V applied to S. Now, going back to $V[H]$, let I be the ideal obtained from I_1. The proof that I is a nonstationary, κ-complete ideal and S is a positive set is as in the above lemma. Let D be the $(\nu+1)$-closed dense subset of I_1^+. Again, as in the above proof, D is still a dense subset of I^+. For the closure, we suppose that $\langle Y_i \mid i < \nu\rangle \in V[H]$ is a decreasing sequence of sets in D. Since $\mathbb{Q}$ is $\leq \nu$-distributive, this sequence is in V, and by closure of D, $\bigcap Y_i \in D$. $\square$

We end the section with the immediate corollary:

Corollary 2.12. *Suppose that $\nu < \kappa$ are cardinals, and $\dagger^\nu_\kappa$ holds in V. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets in V, such that $|\mathbb{P}| \leq \nu$, $\mathbb{P}$ preserves ν, $|\mathbb{Q}| < \kappa$, and $\mathbb{Q}$ is ν^+-closed. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic. Then $\dagger^\nu_\kappa$ still holds in $V[G \times H]$.*

3. Mutual Stationarity and the Tree Property

To obtain the tree property at $\aleph_{\omega+1}$ along with mutual stationarity below $\aleph_\omega$, we use the arguments of [12, Section 3]. The main complication is that to use these techniques, we cannot determine the cardinal that will become $\aleph_1$ in advance.

We will use the following lemma to obtain the tree property.

Lemma 3.1. *[12, Lemma 3.6] Let $\langle \kappa_n \mid n < \omega\rangle$ be a strictly increasing sequence of regular cardinals with supremum ν. Suppose that the following holds:*

- *κ_0 is ν^+-supercompact.*
- *For each $n > 0$, there is a generic ν^+-supercompactness embedding with domain V and critical point κ_n, added by a κ_{n-1}-closed forcing.*

For each strong limit cardinal $\mu < \kappa_0$ with $\operatorname{cof}(\mu) = \omega$, let $\mathbb{L}_\mu$ be the poset $Col(\omega, \mu) \times Col(\mu^+, < \kappa_0)$. Then there is $\mu < \kappa_0$ such that in the extension by $\mathbb{L}_\mu$, the tree property holds at ν^+.

Theorem 3.2. *Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of ν^+-supercompact cardinals, with supremum ν. Then there is a forcing extension in which the tree property holds at $\aleph_{\omega+1}$ and for all $k < \omega$, mutual stationarity holds for $\langle \aleph_n \cap \operatorname{cof}(\omega_k) \mid k < n < \omega\rangle$.*

Proof. Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of supercompact cardinals with supremum ν. Assume also that κ_0 is indestructibly supercompact. Let $\mathbb{H} = \langle \mathbb{H}_n, \dot{\mathbb{H}}(n) \mid n < \omega\rangle$ be the full support iteration where each $\dot{\mathbb{H}}(n)$ is a $\mathbb{H}_n$-name of $Col(\kappa_n, < \kappa_{n+1})$. Let H be generic for $\mathbb{H}$. Note that in $V[H]$, κ_0 remains supercompact and $\kappa_{n+1} = \kappa_n^+$ for all $n < \omega$.

Fix $n < \omega$ and let j be a ν^+ supercompact embedding in V with critical point κ_n. Recall that $\mathbb{H}$ decomposes into $\mathbb{H} = \mathbb{H}_{n-1} * Col(\kappa_{n-1}, < \kappa_n) * (\mathbb{H}/\mathbb{H}_n)$; $\mathbb{H}_{n-1}$ is below the critical point, while $Col(\kappa_{n-1}, < \kappa_n) * (\mathbb{H}/\mathbb{H}_n)$ is κ_{n-1}-closed. Note that the poset $j(\mathbb{H})$ projects to $\mathbb{H}$; this projection is the identity on $\mathbb{H}_{n-1}$, and the induced quotient is κ_{n-1}-closed. It follows that for all $n < \omega$, in $V[H]$ there is a generic ν^+-supercompactness embedding with critical point κ_n, added by a κ_{n-1}-closed forcing.

Applying Lemma 3.1, we see that there exists some strong limit cardinal μ with cofinality ω so that in the extension of $V[H]$ by a generic L for $\mathbb{L}_\mu := Col(\omega,\mu) \times Col(\mu^+, <\kappa_0)$, the tree property holds at ν^+. In $V[H][L]$, $\aleph_n = \kappa_{n+2}$ and $\aleph_\omega = \nu$, so the tree property holds at $\aleph_{\omega+1}$. Note that GCH holds in this model below $\aleph_\omega$.

Remark 3.3. It follows from work of the first author [1] that in this model, the strong tree property holds at $\nu^+ = \aleph_{\omega+1}$. With a slight change, modifying $\mathbb{L}_\mu$ to be $Col(\omega,\mu) \times Col(\mu^{++}, <\kappa_0)$, we could even obtain the super tree property.

Now we turn our attention to proving mutual stationarity in $V[H][L]$. Fix $k < \omega$. We want to show mutual stationarity for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega \rangle$. By Lemma 2.7, since GCH holds in the final model below $\aleph_\omega$, it is sufficient to prove that $\dagger^{\aleph_k}_{\aleph_n}$ holds in $V[H][L]$ for all $n > k+2$; this is accomplished by verifying the hypotheses of Lemmas 2.8 and 2.9.

Fix $n > k$, we claim that $\dagger^{\aleph_k}_{\aleph_{n+2}}$ holds in $V[H][L]$. First note that it is enough to show $\dagger^{\max(\aleph_k,\kappa_0)}_{\aleph_{n+2}}$, as this implies $\dagger^{\aleph_k}_{\aleph_{n+2}}$. Let $\nu = \max(\aleph_k^{V[H][L]}, \kappa_0)$. As $\kappa_n = \aleph_{n+2}^{V[H][L]}$, by Lemma 2.10, it is enough to show $\dagger^\nu_{\kappa_n}$ in $V[H]$.

To that end, in $V[H]$, let $S \subseteq \kappa_n$ be stationary. In V, let j be a κ_{n+1}-supercompactness embedding with critical point κ_n. As before, $j(\mathbb{H})$ projects to $\mathbb{H}$ with a κ_{n-1}-closed quotient. In particular, we meet the hypotheses of Lemma 2.8.

Now consider the decomposition $\mathbb{H} = \mathbb{H}_{n+1} * \mathbb{H}/\mathbb{H}_{n+1}$. Note that the quotient $\mathbb{H}/\mathbb{H}_{n+1}$ is κ_{n+1}-distributive, and so S is a stationary set in $V[\mathbb{H}_{n+1}]$. Since $\mathbb{H}_{n+1}$ is κ_{n+1}-cc, we apply Lemma 2.9 in $V[\mathbb{H}_{n+1}]$ to conclude that there is some condition $r \in \mathbb{H}_{n+1}$ and ordinal γ such that in $V[\mathbb{H}_{n+1}]$, $I_{\gamma,r}$ is nonstationary and $\max(\aleph_k,\kappa_0)+1$-closed and $S \in I^+_{\gamma,r}$. Since the rest of the forcing is $<\kappa_{n+1}$-distributive, $I_{\gamma,r}$ still has the desired properties in the full extension. We have verified that $\dagger^\nu_{\kappa_n}$ holds, completing the proof. □

4. Mutual Stationarity and the failure of SCH

Suppose that in V_0, $\kappa < \mu < \lambda$ are all supercompact cardinals, with κ indestructibly supercompact. Let H be $Col(\kappa,<\mu) * Col(\mu,<\lambda) * Add(\kappa,\lambda)$ -generic and let $V = V_0[H]$.

Lemma 4.1. [2] *There is a normal measure $U^* \in V$ on $\mathcal{P}_\kappa(\lambda^+)$, such that if U_μ is the projected measure to $\mathcal{P}_\kappa(\mu)$, then for every $\gamma < j_{U_\mu}(\kappa)$, $\gamma = j_{U^*}(f)(\kappa)$ for some $f:\kappa\to\kappa$.*

Proof. Let $j_{\lambda^+}: V \to M^*$ be a λ^+-supercompact embedding with critical point κ. Let $j_\mu : V \to M$ be the projected ultrapower to a normal measure on $\mathcal{P}_\kappa(\mu)$. Let $V = \bar{V}[E]$, where E is the $Add(\kappa,\lambda)$-generic. Let $\bar{j}_\mu : \bar{V} \to \bar{M}$ be the restriction of j_μ to $\bar{V}$ and $\bar{j}_{\lambda^+} : \bar{V} \to \bar{M}^*$ be the restriction of j_{λ^+} to $\bar{V}$. Since $|\bar{j}_\mu(\kappa)|^V = |j_\mu(\kappa)|^V = 2^\kappa = \lambda$, enumerate (in V) the interval $[\kappa, \bar{j}_\mu(\kappa)) = \langle u_\alpha \mid \alpha < \lambda\rangle$. We can view the $Add(\kappa,\lambda)$-generic E as a function from $\lambda\times\kappa \to \kappa$, and let $E_\alpha : \kappa\to\kappa$ be $E_\alpha(\delta) = E(\alpha,\delta)$. Let $E^* = j_{\lambda^+}(E)$; a function from $\bar{j}_{\lambda^+}(\lambda) \times \bar{j}_{\lambda^+}(\kappa) \to \bar{j}_{\lambda^+}(\kappa)$. Next we make small changes to E^* to obtain a generic F^* for $\bar{j}_{\lambda^+}(Add(\kappa,\lambda))$. Set F^* to be such that for all $\alpha < \lambda$, $F^*(\bar{j}_{\lambda^+}(\alpha),\kappa) = u_\alpha$, otherwise F^* coincides with E^*. Since the change is captured by a condition, F^* is still generic, and by construction, $\bar{j}_{\lambda^+}$"$E \subset F^*$. So now we can lift $\bar{j}_{\lambda^+}$ to $j'_{\lambda^+} : V = \bar{V}[E] \to \bar{M}^*[F^*]$.

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[2] We do not need this lemma, if we assume a slightly stronger large cardinal hypothesis. See Remark 4.13.

Claim 4.2. *For every u_α, there is a function $f : \kappa \to \kappa$, such that $j'_{\lambda^+}(f)(\kappa) = u_\alpha$.*

Proof. Take $f = E_\alpha$. Then $j'_{\lambda^+}(f)(\kappa) = F^*_{j'_{\lambda^+}(\alpha)}(\kappa) = F^*(j'_{\lambda^+}(\alpha), \kappa) = u_\alpha$. □

We make the analogous change to $j_\mu(E)$ to obtain a generic F for $\bar{j}_\mu(Add(\kappa, \lambda))$, such that for all $\alpha < \lambda$, $F(\bar{j}_\mu(\alpha), \kappa) = u_\alpha$. Here although the change is not quite captured by a condition, all of its initial segments are, so we still have that F is generic. This argument is due to Gitik-Sharon ([8], see Lemma 2.26).

Lift $\bar{j}_\mu$ to j'_μ with respect to F. As in [5, Section 4.1], j'_{λ^+} is obtained by a normal measure U_{λ^+} on $\mathcal{P}_\kappa(\lambda^+)$, and j'_μ is obtained from its projection to a normal measure on $\mathcal{P}_\kappa(\mu)$.

□

Let U_{λ^+} be the normal measure on $\mathcal{P}_\kappa(\lambda^+)$ from the above lemma, and let U_λ be its projection to $\mathcal{P}_\kappa(\lambda)$. Also let U_μ be its projection to $\mathcal{P}_\kappa(\mu)$ and let U be its projection to a normal measure on κ. Set $j_{\lambda^+} := j_{U_{\lambda^+}} : V \to M_{\lambda^+}$, $j_\lambda := j_{U_\lambda} : V \to M_\lambda$, $j_\mu := j_{U_\mu} : V \to M_\mu$, and $j := j_U : V \to M$.

Let $k : M \to M_\mu$ be $k([f]_U) = j_\mu(f)(\kappa)$. Then $j_\mu = k \circ j$, and by construction each u_α is in the range of k. It follows that $crit(k) \geq j_\mu(\kappa)$. And actually, since $j_\mu(\kappa)$ is also in the range of k, $crit(k) > j_\mu(\kappa)$, and so $j(\kappa) = j_\mu(\kappa)$.

Similarly, let $k^* : M \to M_{\lambda^+}$ be $k^*([f]_U) = j_{\lambda^+}(f)(\kappa)$. Then $j_{\lambda^+} = k^* \circ j$, and by construction each u_α is in the range of k^*. It follows that $crit(k^*) \geq j_\mu(\kappa) = j(\kappa)$. Since $|j(\kappa)|^V = \lambda < |j_{\lambda^+}(\kappa)|^V = \lambda^{++}$, we must have $crit(k^*) = j(\kappa)$.

Let $\mathbb{P}$ be the Prikry forcing with respect to U with interleaved collapses and guiding generics to make $\kappa = \aleph_\omega$ and preserve cardinals above κ. More precisely, conditions in $\mathbb{P}$ are of the form $p = \langle d, \alpha_0, c_0, ..., \alpha_{n-1}, c_{n-1}, A, C\rangle$, where $lh(p) = n$ and:

(1) $\langle \alpha_i \mid i < n\rangle$ is an increasing sequence in κ, $A \in U$;
(2) $d \in Col(\omega_1, < \alpha_0)$ if $n > 0$; otherwise $d \in Col(\omega_1, < \kappa)$.
(3) $c_i \in Col(\alpha_i^{++}, < \alpha_{i+1})$ if $i < n-1$, and $c_{n-1} \in Col(\alpha_{n-1}^{++}, < \kappa)$;
(4) $\mathrm{dom}(C) = A$, for each $\alpha \in A$, $C(\alpha) \in Col(\alpha^{++}, < \kappa)$, $[C] \in K$, where K is a guiding generic for $Col(\kappa^{++}, < j(\kappa))^{Ult(V,U)}$.

Let us briefly describe how we get K. The number of antichains in $\mathbb{C} := Col(\kappa^{++}, < j(\kappa))^{Ult(V,U)}$ is κ^{++}; enumerate them by $\langle A_i \mid i < \kappa^{++}\rangle$. By the high critical point of k, we have that $k(\mathbb{C}) = \mathbb{C}$ and for each i, $k(A_i) = k" A_i = A_i$. So working in M_μ, which is closed under sequences of length κ^+, and satisfies that $\mathbb{C}$ is $< \kappa^{++}$-closed, build a decreasing sequence of conditions meeting these antichains. Then use them to define K.

Let G be $\mathbb{P}$-generic. We have the following standard properties about $V[G]$:

(1) κ is preserved by the Prikry lemma, and becomes $\aleph_\omega$.
(2) $\mathbb{P}$ has the κ^+ chain condition, so cardinals above κ are preserved, and $2^{\aleph_\omega} = \lambda = \aleph_{\omega+2}$.
(3) G adds a Prikry sequence $\langle \kappa_n \mid n < \omega\rangle$, with limit κ, such that for all $A \in U$, for all large n, $\kappa_n \in A$;
(4) G adds a sequence $\langle c^*_n \mid n < \omega\rangle$, such that each c^*_n is generic for $Col^V(\kappa_n^{++}, < \kappa_{n+1})$.

We will show that $V[G]$ is the desired model for theorem 1.3. To do that we will show that in $V[G]$, $\dagger^{\aleph_k}_{\aleph_n}$ holds for all $k > 0$ and all large $n > k$ and that all relevant stationary sets consists of approachable points. We only have to worry about cardinals of one of the following three types: κ_n, κ_n^+, and κ_n^{++} for $n < \omega$, as the other cardinals below κ are collapsed.

Fix $k > 0$, and by increasing k if necessary, we may assume that $\aleph_k$ is a Prikry point, say $\kappa_{\bar{n}}$. (Here we use that if $\nu' > \nu$, then $\dagger^{\nu'}_\kappa$ implies $\dagger^\nu_\kappa$.) Let $\nu < \kappa$ be such that some condition in $\mathbb{P}$ forces that $\nu = \dot{\aleph}_k = \dot{\kappa}_{\bar{n}}$. For the rest of the section, whenever we talk about $V[G]$ assume we are working below this condition. We will show that in $V[G]$, for all large n, we have $\dagger^\nu_{\kappa_n}$, $\dagger^\nu_{\kappa_n^+}$, and $\dagger^\nu_{\kappa_n^{++}}$.

4.1. **The Prikry points.**

Lemma 4.3. *In V, for all regular τ with $\nu < \tau < \kappa$, we have that $\dagger^\nu_{\kappa, Col(\tau^{++}, <\kappa)}$ holds. Moreover, there is a measure one set $A_\tau \in U$, such that for all $\alpha \in A_\tau$, $\dagger^\nu_{\alpha, Col(\tau^{++}, <\alpha)}$ holds.*

Proof. Note that $\dagger^\nu_{\kappa, Col(\tau^{++}, <\kappa)}$ asserts the existence of certain ideals on κ, which are subsets of 2^κ. We will construct these ideals from the supercompactness of κ, using Lemma 2.8.

Fix τ. Recall that $j_\lambda : V \to M_\lambda$ is the λ-supercompactness embedding with critical point κ, projecting to U. I.e. $U = \{A \mid \kappa \in j_\lambda(A)\}$ is the normal measure used in the definition of the Prikry forcing. We have that there exists a projection from $j_\lambda(Col(\tau^{++}, < \kappa))$ onto $Col(\tau^{++}, < \kappa)$ with a τ^{++}-closed quotient, so by Lemma 2.8 every ideal $I_{\gamma,r}$ will be $(\nu + 1)$-closed. It remains to verify that for any name for a stationary set $\dot{S}$, there is some choice of (γ, r) such that the ideal $I_{\gamma,r}$ is nonstationary and $\dot{S}$ is a positive set with respect to this ideal. This follows from Lemma 2.9, noting that $Col(\tau^{++}, < \kappa)$ is κ-cc. So, $\dagger^\nu_{\kappa, Col(\tau^{++}, <\kappa)}$ holds in V.

Since $M_\lambda^\lambda = M_\lambda^{2^\kappa} \subseteq M_\lambda$, in M_λ, $\dagger^\nu_{\kappa, Col(\tau^{++}, <\kappa)}$ also holds. It follows that for U-many α, $\dagger^\nu_{\alpha, Col(\tau^{++}, <\alpha)}$ holds in V. □

Now, let A_τ be given by the above lemma for each $\tau > \nu$ and set $A^* = \triangle_{\tau < \kappa} A_\tau$. By forcing below A^*, we may assume that each Prikry point $\kappa_n \in A^*$.

Lemma 4.4. *For all large n, in V, $\dagger^\nu_{\kappa_n, Col(\kappa_{n-1}^{++}, <\kappa_n)}$ holds.*

Proof. Fix n such that it is forced that $\nu < \kappa_{n-1}$. By choice of A^*, we have that for all τ, for all $\alpha \in A^* \setminus (\tau + 1)$, $\dagger^\nu_{\alpha, Col(\tau^{++}, <\alpha)}$ holds in V. In particular, for all $\alpha \in A^*$ with $\alpha > \kappa_{n-1}$, $\dagger^\nu_{\alpha, Col(\kappa_{n-1}^{++}, <\alpha)}$ holds in V. Since $\kappa_n \in A^*$ with $\kappa_n > \kappa_{n-1}$, we have that $\dagger^\nu_{\kappa_n, Col(\kappa_{n-1}^{++}, <\kappa_n)}$ holds in V. □

As a corollary, by definition of $\dagger$, we have that:

Lemma 4.5. *For all large n, $\dagger^\nu_{\kappa_n}$ holds in $V[c^*_{n-1}]$.*

Lemma 4.6. *For all large n, $\dagger^\nu_{\kappa_n}$ holds in $V[\langle c^*_i \mid i < n \rangle]$.*

Proof. We have $V[\langle c^*_i \mid i < n \rangle] = V[c^*_{n-1}][\langle c^*_i \mid i < n - 1 \rangle]$, and we can factor $\langle c^*_i \mid i < n - 1 \rangle = \langle c^*_i \mid i < \bar{n} \rangle \times \langle c^*_i \mid \bar{n} < i < n - 1 \rangle$. The first factor is of size $\kappa_{\bar{n}}$ and the second factor is $\kappa_{\bar{n}}^{++}$-closed and has size κ_{n-1}, where recall that $\kappa_{\bar{n}} = \nu$. Then, by corollary 2.12, we have that $\dagger^\nu_{\kappa_n}$ still holds in $V[\langle c^*_i \mid i < n \rangle]$.

□

Lemma 4.7. *In $V[G]$, for all large n, $\dagger^\nu_{\kappa_n}$ holds.*

Proof. First note that $V[G]$ projects to $V[\langle c^*_i \mid i \leq n\rangle]$ by a quotient that does not add subsets of κ_{n+1} (this is [11, Theorem 3.2]), and $\dagger^\nu_{\kappa_n}$ is a statement about subsets of $\mathcal{P}(\kappa_n)$. So if $\dagger^\nu_{\kappa_n}$ holds in $V[\langle c^*_i \mid i \leq n\rangle]$, then in also holds in $V[G]$.

Next we show that $\dagger^\nu_{\kappa_n}$ holds in $V[\langle c^*_i \mid i \leq n\rangle]$. Suppose that $S \subset \kappa_n$ is a stationary set in $V[\langle c^*_i \mid i \leq n\rangle] = V[\langle c^*_i \mid i < n\rangle][c^*_n]$. Here c^*_n is generic for $Col(\kappa_n^{++}, < \kappa_{n+1})$, and so $S \in V[\langle c^*_i \mid i < n\rangle]$. Let $I \in V[\langle c^*_i \mid i < n\rangle]$ be the nonstationary κ_n-complete, $\nu + 1$-closed ideal on κ_n, with $S \in I^+$, given by $\dagger^\nu_{\kappa_n}$ in that model. Since $Col(\kappa_n^{++}, < \kappa_{n+1})$ does not add new subsets of κ_n, I is still a non stationary ideal in the bigger model $V[\langle c^*_i \mid i < n\rangle][c^*_n]$. Moreover, since $Col(\kappa_n^{++}, < \kappa_{n+1})$ is κ_n^{++}-closed, I is still κ_n-complete and $\nu + 1$-closed. So I is as desired. □

4.2. **The first successors,** κ_n^+**.**

Lemma 4.8. *In $V[G]$, we have that for all large n, $\dagger^\nu_{\kappa_n^+}$ holds.*

Proof. Note that $2^{\kappa_n^+} = \kappa_n^{++}$. Since the quotient to get from $V[G]$ from $V[\langle c^*_i \mid i \leq n\rangle]$ does not add subsets of κ_{n+1}, it is enough to show that $\dagger^\nu_{\kappa_n^+}$ holds in $V[\langle c^*_i \mid i \leq n\rangle]$. Also, $\langle c^*_i \mid i \leq n-1\rangle$ is a generic for a forcing of the form $\mathbb{P}_1 \times \mathbb{P}_2$, where $\mathbb{P}_1$ has size $\kappa_{\bar{n}}$ and $\mathbb{P}_2$ is $\kappa_{\bar{n}}^{++}$-closed and has size κ_n. Recalling that $\nu = \kappa_{\bar{n}}$, by Corollary 2.12, it is enough to show $\dagger^\nu_{\kappa_n^+}$ holds in $V[c^*_n] = V[Col(\kappa_n^{++}, < \kappa_{n+1})]$.

Claim 4.9. *For all large n, $\dagger^\nu_{\kappa_n^+}$ holds in V.*

Proof. Recall that V is the extension of V_0 by the poset $Col(\kappa, < \mu) * \dot{C}ol(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda)$. Let $i : V_0 \to M_0$ be a $2^\mu = \lambda$-supercompactness embedding with critical point μ. Note that $i(Col(\kappa, < \mu) * \dot{C}ol(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda))$ absorbs $Col(\kappa, < \mu) * \dot{C}ol(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda)$ and the quotient is κ-closed.

By Lemmas 2.8 and 2.9, noting that $Col(\kappa, < \mu) * \dot{C}ol(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda)$ is λ-cc and κ-closed, we conclude that $\dagger^\nu_\mu$ holds in V.

Now we use the λ-supercompactness embedding with critical point κ, $j_\lambda : V \to M_\lambda$. Since $2^\mu = \lambda$ and $M_\lambda^\lambda \subset M_\lambda$, we also have that $\dagger^\nu_\mu$ (i.e. $\dagger^\nu_{\kappa^+}$) holds in M_λ. Then there is a measure one set $A \in U$ such that for all $\alpha \in A$, $\dagger^\nu_{\alpha^+}$ holds in V. It follows that for all large n, $\kappa_n \in A$, and so $\dagger^\nu_{\kappa_n^+}$ holds in V. □

Claim 4.10. *For all large n, $\dagger^\nu_{\kappa_n^+}$ holds in $V[c^*_n]$.*

Proof. Let n be such that $\dagger^\nu_{\kappa_n^+}$ holds in V. Suppose that $S \subset \kappa_n^+$ be a stationary set in $V[c^*_n]$. Since $Col(\kappa_n^{++}, < \kappa_{n+1})$ does not add any subsets of κ_n^+, S is a stationary set in V. Let $I \in V$ be a nonstationary, κ_n^+-complete, $(\nu + 1)$-closed ideal on κ_n^+ with $S \in I^+$, given by $\dagger^\nu_{\kappa_n^+}$ in V. Since $Col(\kappa_n^{++}, < \kappa_{n+1})$ is κ_n^{++}-closed, I remains a nonstationary, κ_n^+-complete, $(\nu + 1)$-closed ideal in $V[c^*_n]$. □

□

4.3. **The second successors,** κ_n^{++}**.**

Lemma 4.11. *In V, $\dagger^\nu_{\lambda, Col(\lambda, <\tau)}$ holds for all $\tau > \lambda$.*

Proof. Let $\tau > \lambda$. Let V' be a generic extension of V_0 by $Col(\kappa, < \mu)$, and let $j : V' \to M$ be a τ-supercompact embedding with critical point λ. Since $j(Col(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda) * \dot{C}ol(\lambda, < \tau))$ projects to $Col(\mu, < \lambda) * \dot{A}dd(\kappa, \lambda) * \dot{C}ol(\lambda, < \tau)$, we can lift j to $j : V'[Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda) * \dot{Col}(\lambda, <\tau)] \to M^*$. Moreover, $\mathbb{P} := Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda) * \dot{Col}(\lambda, <\tau)$ is τ-c.c.

Let S be stationary in $V'[Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda) * \dot{Col}(\lambda, <\tau)]$. By Lemma 2.9 that there is some condition $r \in j(\mathbb{P})/\mathbb{P}$ and ordinal $\gamma \in j(\kappa) \setminus \kappa$ so that the ideal $I_{\gamma,r}$ is nonstationary and $S \in I^+_{\gamma,r}$. Note also that the quotient $j(Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda) * \dot{Col}(\lambda, <\tau))/(Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda) * \dot{Col}(\lambda, <\tau))$ is κ-closed. Since $\nu < \kappa$, from Lemma 2.8, we can conclude that $I_{\gamma,r}$ is λ-complete, nonprincipal, and $(\nu + 1)$-closed.

Since V is the extension of V' by $Col(\mu, <\lambda) * \dot{Add}(\kappa, \lambda)$, we conclude that in V, $\dagger^\nu_{\lambda, Col(\lambda, <\tau)}$ holds. □

Remark 4.12. By the same argument as above, we can get $\dagger^\nu_{\lambda, Col(\lambda, <\gamma)}$ in V even if γ is not a cardinal. We just have to use a $|\gamma|^+$-supercompact embedding with critical point λ.

Remark 4.13. Next we will use Lemma 4.1. We note that we do not need it if we assume a slightly stronger large cardinal hypothesis that there is a normal measure on $\mathcal{P}_\kappa(\lambda)$, such that for measure one many $\tau < \kappa$, τ is $< j(\kappa)$-supercompact in the ultrapower.

Lemma 4.14. *For all large $n < \omega$, in $V[G]$ we have that $\dagger^\nu_{\kappa_n^{++}}$ holds.*

Proof. As before, it is enough to show that $\dagger^\nu_{\kappa_n^{++}}$ holds in $V[c^*_n]$.

Recall that we chose a λ^+-supercompact embedding with critical point κ, $j_{\lambda^+} : V \to M_{\lambda^+}$, so that the corresponding $k^* : M \to M_{\lambda^+}$ has critical point $j(\kappa)$. (Here $M = Ult(V, U)$ where U is the projected normal measure on κ, used in the definition of the Prikry forcing).

Claim 4.15. *There is a measure one set $A \in U$ such that for all $\alpha \in A$ and all τ with $\alpha^{++} < \tau < \kappa$ we have $\dagger^\nu_{\alpha^{++}, Col(\alpha^{++}, <\tau)}$ holds in V.*

Proof. Let $\lambda < \gamma < j(\kappa)$, γ a cardinal in M. By Lemma 4.11 and the subsequent remark, we have that $\dagger^\nu_{\lambda, Col(\lambda, <\gamma)}$ holds in V. Since $|\gamma|^V \leq \lambda$, $2^\lambda = \lambda^+$, and $(M_{\lambda^+})^{\lambda^+} \subset M_{\lambda^+}$, we also have that, $\dagger^\nu_{\lambda, Col(\lambda, <\gamma)}$ holds in M_{λ^+}.

By the high critical point of k^*, $k^*(\gamma) = \gamma$, so by the elementarity of k^*, $M \models \dagger^\nu_{\lambda, Col(\lambda, <\gamma)}$.

We have shown that in M, for all τ with $\lambda < \tau < j(\kappa)$, $\dagger^\nu_{\lambda, Col(\lambda, <\tau)}$ holds. So there is $A \in U$, such that for all $\alpha \in A$, and all τ with $\alpha^{++} < \tau < \kappa$ we have $\dagger^\nu_{\alpha^{++}, Col(\alpha^{++}, <\tau)}$ holds in V. □

It follows from the claim that for all large n, $V \models \dagger^\nu_{\kappa_n^{++}, Col(\kappa_n^{++}, <\kappa_{n+1})}$. So for all large n, $\dagger^\nu_{\kappa_n^{++}}$ holds in $V[c^*_n]$. □

4.4. **Mutual stationarity in the final model.** We can finally prove the main theorem of the section:

Theorem 4.16. *In $V[G]$, we have the failure of SCH at $\aleph_\omega$ and mutual stationarity for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega \rangle$ for every $k < \omega$.*

Proof. Clearly SCH at $\aleph_\omega$ fails. Fix $k < \omega$. It is a well-known fact due to Shelah [13] that for all $n > k + 1$, $\aleph_n \cap \mathrm{cof}(\aleph_k)$ is approachable. Mutual stationarity follows since in $V[G]$, we have $\dagger^{\aleph_k}_{\aleph_n}$ for all large n. □

We end with the following open questions:

Question. Do the analogues of our two main theorems hold for singular cardinals of uncountable cofinality? In particular, for any countable ρ, can we obtain mutual stationarity for $\langle \aleph_\eta \cap \mathrm{cof}(\aleph_{\rho+1}) \mid \rho + 1 < \eta < \omega_1 \rangle$ together with the failure of SCH at $\aleph_{\omega_1}$? What about together with the tree property at $\aleph_{\omega_1+1}$?

Question. Can we obtain a model where mutual stationarity for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega \rangle$ holds together with reflection at $\aleph_{\omega+1}$ and the failure of SCH at $\aleph_\omega$?

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